Developing the concept of the limit of a function

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Consider the following definition:

$$\lim_{x \to a} f(x) = L \text{ means that if x is close to a then f(x) is close to L.}$$

Suppose we examine this proposed definition to see to what extent it does what we want it to do There is a serious defect in this attempt to define $\lim_{x \to a} f(x) = L$. This defect will be illustrated by the following example. Consider a set of functions $f_1, f_2, f_3, \ldots$, where the $f_n(x) = 10^n + 1$, for n = 1, 2, 3, ... For **each n** the true limit is given by the following. $\lim_{x \to 0} (10^n x + 1) = 1$ Suppose now we agree that $x_0 = 0.000001$ is close to 0. From the definition above it would follow that $f_n(x_0)$ is close to L = 1. We check this for a few values of n:

$f_1(x_0) = 10\, x_0 + 1 = 1.00001,$

$f_2(x_0) = 10^2 x_0 + 1 = 1.0001,$

$f_3(x_0) = 10^3 x_0 + 1 = 1.001,$

$f_4(x_0) = 10^4 x_0 + 1 = 1.01,$

We probably would not rebel against the statement that these numbers are "close to 1." Nor would we likely object to saying that $f_5(x_0) = 10^5 x_0 + 1 = 1.1$ and $f_6(x_0) = 10^6 x_0 + 1 = 2$ are "close to 1." But if we continue in this manner we soon find the corresponding statements not only unpalatable but downright unacceptable:

$f_7(x_0) = 10^7 x_0 + 1 = 11,$

$f_8(x_0) = 10^8 x_0 + 1 = 101,$

$f_9(x_0) = 10^9 x_0 + 1 = 1001,$

$f_{10}(x_0) = 10^{10}\, x_0 + 1 = 10001,$

This should be enough to make our point. We may be willing to agree that 1.0001 is close to 1, but not that 10, 001 is close to 1.

And yet we still insist that for *every n*, n = 1, 2, 3, ..., $\lim_{x \to 0} (10^n x + 1) = 1$ is correct. (The graph of each of these functions is a straight line with y-intercept equal to 1, i.e., when x = 0, y = 1.) *Can* we make $f_{10}(x_0) = 10^{10}\, x_0 + 1 = 10001$ "close to 1"? If we are still willing to agree that 1.0001 is close to 1, it is easy to see that all we need do is choose $x_1 = 10^{-14}$: then $f_{10}(x_1) = 1.0001$. Or, if one demands that nothing larger than 1.00000001 can be called close to 1, then that one can be silenced by being presented with $x_2 = 10^{-18}$: $f_{10}(x_2) = 1.00000001$.

Thus a careful examination of the nature of the defect discloses a remedy: instead of *starting* with an x "close to a," we begin by specifying *how close we want f(x) to be to L,* and then inquiring whether or not an x satisfying this requirement can be found.

We now use this suggestion to revise our definition.

$\lim\limits_{x \to a} f(x) = L$ means the numbers f(x) can be made to lie as close to L as we please by taking x close enough to a.

This form does repair the flaw discussed above. Unfortunately this definition also has a defect. Consider the following piecewise function:

$$f(x) = \begin{cases} x + 1 & \text{if } x \text{ is rational} \\ x + 3 & \text{if } x \text{ is irrational} \end{cases}$$

Now if we assert that $\lim\limits_{x \to 0} f(x) = 1$, then we can certainly make the f(x) values lie as close to 1 as we please by taking x sufficiently close to 0, *provided we take rational numbers for x.* Similarly, we could assert that $\lim\limits_{x \to 0} f(x) = 3$ and argue that the f(x) numbers can be made to lie as close to 3 as we please by taking x sufficiently close to 0, *provided we are allowed to use irrational x.*

We conclude that this second attempt does not have uniqueness. But once again the illustrated difficulty suggests a way to remove our insistence that we be able to choose x in special ways. If we remove this possibility the trouble presented by this example will go away.

This we now do.

$\lim\limits_{x \to a} f(x) = L$ means the numbers f(x) will lie as close to L as we please for all x sufficiently close to a.

A little reflection will show that , according to this definition, we can no longer say for the function of the preceding example that $\lim\limits_{x \to 0} f(x) = 1$ or $\lim\limits_{x \to 0} f(x) = 3$, or, in fact, $\lim\limits_{x \to 0} f(x) = L$, no matter what L we choose. Have we achieved our aim? Again we cannot be content. For if our aim is to obtain a *precise* definition, we must readily admit that a definition which includes such phrases as “as close to L as we please” and “sufficiently close to a” leaves something to be desired. We have already observed that the question of when a number is to be considered close to 1 can be answered in various ways. In order to sharpen the definition, we introduce a simple concept, that of *neighborhood* (abbreviated nbd.) of a number.

By a **nbd. of a, $N_\delta(a)$**, we mean an interval with its center at a. More precisely,

$$N_\delta(a) = \{x \mid a - \delta < x < a + \delta\} = \{x \mid |x - a| < \delta\}$$

We can now rephrase the current definition of limit as follows:

$\lim\limits_{x \to a} f(x) = L$ means that given any $N_\epsilon(L)$, no matter how small $\epsilon > 0$ is, we can find a $N_\delta(a)$, $\delta > 0$, such that $x \in N_\delta(a) \Rightarrow f(x) \in N_\epsilon(L)$.

This simply says that no matter how close we want f(x) to be to L (this is what the $N_\epsilon(L)$ prescribes) we can get x close enough to a (this is what $N_\delta(a)$ gives) to guarantee that f(x) will be that close to L. It would seem as if we have surely achieved our purpose and that this definition can be taken as the final version. Not so. This definition does not enable us to deal with the limits like $\lim\limits_{x \to 0} \frac{x}{x}$.

We begin with three observations:

1. If $x \neq 0$, then $\frac{x}{x} = 1$.
2. We cannot have x = 0.
3. This example looks artificial, but it is in fact a simple example of what must be faced up to in calculus (namely, the "zero over zero" kind of limit).

Notice that, excluding x = 0 from consideration, in any N(0) we have $\frac{x}{x}$ equals 1 – not only *is close to* 1, but *equals* 1. We cannot apply the current definition to this limit, because $0 \in N(0)$, whereas $0 \notin$ domain (f) where $f(x) = \frac{x}{x}$. Once again the nature of the difficulty suggests a simple remedy: namely exclude 0 from N(0). This may seem a little unexpected and arbitrary, but, as we shall see, it is a device which works. We begin by modifying the definition of nbd.

By a d**eleted nbd. of a**, $N_\delta^*(a)$, we mean exactly $N_\delta(a)$ with the number a deleted:

$$N_\delta^*(a) = \{x \mid 0 < | x - a | < \delta \} = x \mid a - \delta < x < a + \delta, x \neq 0 \}$$

We are now ready to give the FINAL version of our definition.

$\lim\limits_{x\to a}$ f(x) = L means that $N_\epsilon(L)$, no matter how small $\epsilon > 0$ is, there exist a $N_\delta^*(a)$, $\delta > 0$, such that for every $x \in N_\delta^*(a) \Rightarrow f(x) \in N_\epsilon(L)$.

Notice that a deleted nbd. of a is used, but that the nbd. of L still contains L.

There are several alternative statements for this final definition of limit.

$\lim\limits_{x\to a}$ f(x) = L means that for every $\epsilon > 0$, , no matter how small, there exist a $\delta_\epsilon > 0$, such that for all x satisfying $0 < |x - a| < \delta_\epsilon$ it is true that f(x) satisfies $|f(x) - L | < \epsilon$.

Another form is given by:

$\lim\limits_{x\to a}$ f(x) = L means that $\forall N_\epsilon(L)$, $\exists N_\delta^*(a) \ni \forall x \in N_\delta^*(a)$ then $f(x) \in N_\epsilon(L)$.

It was an interesting journey!!